# Riesz endomorphisms of Banach algebras

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**Abstract**

Let $B$ be a unital commutative semi-simple Banach algebra. We study endomorphisms of $B$ which are simultaneously Riesz operators. Clearly compact and power compact endomorphisms are Riesz. Several general theorems about Riesz endomorphisms are proved, and these results are then applied to the question of when Riesz endomorphisms of certain algebras are necessarily power compact.

## Introduction

Let $B$ be a unital commutative semi-simple Banach algebra. An endomorphism $T : B \to B$ is a linear operator which also preserves multiplication. An endomorphism $T$ is called unital if $T1 = 1$. Since the Banach algebra is assumed to be semi-simple, it follows from very early theory that $T$ is necessarily bounded. It is interesting to see what can be deduced if various operator theoretic properties are imposed on the endomorphism. In this note we consider endomorphisms which are also Riesz operators. We will start out by proving two theorems which are valid for all unital commutative semi-simple Banach algebras, and then apply them to specific examples.

If $B$ is a Banach space, a Riesz operator $T : B \to B$ is a bounded linear map satisfying properties (a) - (e) [11].

(a) For each non-zero $\lambda$, $\lambda - T$ is open.

(b) For each non-zero $\lambda$, $(\lambda - T)(B)$ is closed.

If $S$ is a linear operator $B \to B$, let $N(S) = \{f : Sf = 0\}$, and $R(S) = S(B)$ the range of $S$.

(c) For each non-zero $\lambda$, $\dim N(\lambda - T)$ and $\mathrm{codim} R(\lambda - T)$ are finite and equal.

(d) For each non-zero $\lambda$, the lengths of the null chain $N(\lambda - T) \subseteq N((\lambda - T)^2) \subseteq \cdots$, and the image chain $R(\lambda - T) \supseteq R((\lambda - T)^2) \cdots$ are finite and equal.

(e) The non-zero spectrum of $T$ consists of eigenvalues, and if there are infinitely many of them, they form a sequence approaching 0.

Proposition 52.2 in [11] contains the following useful characterization of Riesz operators.

**Theorem:** Let $B$ be a Banach space and $T$ a bounded linear operator from $B$ to $B$. Then $T$ is a Riesz operator if, and only if,

$\lim[\inf\{\|T^n - K\| : K \text{ is a compact operator } B \to B\}]^{1/n} = 0.$

The limit on the left hand side of the last equation is called the essential spectral radius of $T$. Clearly compact operators and quasinilpotent operators are Riesz operators. The sum of a compact operator and a quasinilpotent operator is Riesz. A bounded linear operator $T$ is power compact if for some positive integer $N$, $T^N$ is compact. Power compact operators are Riesz. One of the questions considered is for which algebras is every Riesz endomorphism necessarily power compact.

## Part 1

If $B$ is a unital commutative simi-simple Banach algebra with maximal ideal space $X$, and $T$ is a unital endomorphism of $B$, then it is well known that there exists a w*-continuous selfmap $\phi$ of $X$ such that for all $f \in B$ and $x \in X$, $\widehat{Tf}(x) = \hat{f}(\phi(x))$. In this case we say that $T$ is induced by $\phi$ or that $\phi$ induces $T$.

It was shown in [12] that if $\phi$ induces a compact endomorphism of the unital commutative simi-simple Banach algebra $B$, and the maximal ideal space $X$ of $B$ is connected, then $\bigcap_{n=0}^{\infty} \phi_n(X) = \{x_0\}$ for some $x_0 \in X$. Here $\phi_n$ denotes the nth iterate of $\phi$.

The results in [12] included not necessarily unital algebras and not necessarily connected maximal ideal spaces. However, if we specialize to our case (unital $B$, connected $X$), then the key elements in the proof of Theorem 1.7 of [12] are Lemmas 1.4 and 1.6 of that paper. It is not hard to see that the proof of Lemma 1.4 depends only on the fact that non-zero elements in the spectrum of $T$ are eigenvalues of finite multiplicity, while properties (a)-(d) of the definition of Riesz operators are all that are needed to prove Lemma 1.6. Thus we can state the following theorem.

**Theorem 1.1:** If $B$ is a unital commutative semi-simple Banach algebra with connected maximal ideal space $X$, and if $T$ is a Riesz endomorphism of $B$ induced by a selfmap $\phi$ of $X$, then $\bigcap_{n=0}^{\infty} \phi_n(X) = \{x_0\}$ where $x_0 \in X$ is a fixed point of $\phi$ which is unique.

If $B$ is a unital commutative semi-simple Banach algebra with maximal

ideal space $X$, for $x, y \in X$ we let

$$\|x - y\| = \sup\{|\hat{f}(x) - \hat{f}(y)| : f \in B, \|f\| \leq 1\}.$$

That is, $\|x - y\|$ is the norm of $x - y$ regarded as an element of the dual space $B^*$ of $B$. Further, for $\varepsilon > 0$ and $a \in X$, we let $B(a, \varepsilon) = \{x \in X : \|x - a\| < \varepsilon\}$.

We now prove the following theorem for arbitrary unital commutative semi-simple Banach algebras. We remark that a related argument was used by L. Zheng ([15] Lemma 2) in connection with Riesz composition operators on $H^\infty$ of the unit disk.

**Theorem 1.2:** Suppose that $B$ is a unital commutative semi-simple Banach with connected maximal ideal space $X$. Let $T$ be a Riesz endomorphism induced by a selfmap $\phi$ of $X$. Suppose that $\{x_0\} = \bigcap_{n=0}^\infty \phi_n(X)$. Then for each $\varepsilon > 0$ there exists a positive integer $N$ such that $\phi_N(X) \subset B(x_0, \varepsilon)$.

**Proof:** Assume that $B, X, T, \phi, x_0$ are as described in the hypothesis. Suppose that there exists $\varepsilon > 0$ such that for all positive integers $k$ we have that $\phi_k(X)$ is not contained in $B(x_0, \varepsilon)$. We show that this leads to a contradiction that $T$ is a Riesz operator. To this end, let $x_n \in X$ satisfy $\phi_n(x_n) \notin B(x_0, \varepsilon)$, i.e. $\|\phi_n(x_n) - x_0\| \geq \varepsilon$. From Theorem 1.1 , if $\mathcal{U}$ is a w*-open neighborhood of $x_0$, then $\phi_n(X) \subset \mathcal{U}$ for large $n$. In particular, $\phi_n(x_n) \in \mathcal{U}$ for large $n$, and so it follows that $\phi(x_n) \to x_0$ in the w*-topology of $X$.

Let $y_n = \phi_{n-1}(x_n)$. Then $y_n \to x_0$ in the w*-topology of $X$ and $\|\phi(y_n) - x_0\| \geq \varepsilon$. Choose $f_n \in B$ with $\|f_n\|_B = 1$ and $|\hat{f}_n(\phi(y_n)) - \hat{f}_n(x_0)| > \varepsilon - \frac{1}{n}$. Let $K$ be any compact linear map from $B \to B$. Then there exist $g \in B$ and a subsequence $\{f_{n_j}\}$ with $Kf_{n_j} \to g$ in norm.

Hence

$$\|T - K\| \geq \|Tf_{n_j} - Kf_{n_j}\| \geq \|Tf_{n_j} - g\| - \|Kf_{n_j} - g\|$$

$$\geq |\hat{f}_{n_j}(\phi(y_{n_j})) - \hat{g}(y_{n_j})| - \|Kf_{n_j} - g\|.$$

Therefore

$$\|T - K\| \geq \limsup |\hat{f}_{n_j}(\phi(y_{n_j})) - \hat{g}(y_{n_j})| = \limsup |\hat{f}_{n_j}(\phi(y_{n_j})) - \hat{g}(x_0)|.$$

Also, evaluating at $x = x_0$ gives that

$$\|T - K\| \geq \limsup[|\hat{f}_{n_j}(\phi(x_0)) - \hat{g}(x_0)| - \|Kf_{n_j} - g\|] = \limsup |\hat{f}_{n_j}(x_0) - \hat{g}(x_0)|.$$

Adding, we obtain

$$2\|T - K\| \geq \limsup[|\hat{f}_{n_j}(\phi(y_{n_j}) - \hat{g}(x_0)| + |\hat{f}_{n_j}(x_0) - \hat{g}(x_0)|],$$

and so

$$2\|T-K\| \geq \limsup |\hat{f}_{n_j}(\phi(y_{n_j})) - \hat{f}_{n_j}(x_0)| \geq \varepsilon.$$

Therefore, $\|T-K\| \geq \varepsilon/2$ for all compact operators $K$.

Next for each positive integer $m$, consider the endomorphism $T^m$ which is induced by $\phi_m$. Then $T^m$ is Riesz and we have $\bigcap \phi_{mn}(X) = \{x_0\}$. If we apply the previous argument to $T^m$, we get that $\|T^m - K\| \geq \varepsilon/2$ for each positive integer $m$ and all compact operators $K$. Therefore $\lim_m (\inf\{\|T^m - K\| : K$ is compact$\})^{1/m} \geq 1$. This contradicts the assumption that $T$ is a Riesz operator. Hence we have that if $T$ is a Riesz endomorphism induced by $\phi$, then for each $\varepsilon > 0$ there exists a positive integer $N$ with $\phi_N(X) \subset B(x_0, \varepsilon)$. $\square$

There are several immediate consequences of this theorem.

**Corollary 1.3:** Suppose that $B$ is a unital commutative semi-simple Banach with connected maximal ideal space $X$. Let $T$ be a Riesz endomorphism induced by a selfmap $\phi$ of $X$. Suppose that $\{x_0\} = \bigcap_{n=0}^{\infty} \phi_n(X)$. If $x_0$ is an isolated point of $X$ in the norm topology, then $T^N f = \hat{f}(x_0)1$ for some positive integer $N$. $\square$

**Proof:** Since $x_0$ is an isolated point in the norm topology, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) = \{x_0\}$ and the result follows from Theorem 1.2. $\square$

**Corollary 1.4:** Suppose that $B$ is a unital commutative semi-simple Banach with connected maximal ideal space $X$. Let $T$ be a Riesz endomorphism induced by a selfmap $\phi$ of $X$. Suppose that $\{x_0\} = \bigcap_{n=0}^{\infty} \phi_n(X)$. If $B$ has no non-zero point derivations at $x_0$, then $T^N f = \hat{f}(x_0)1$ for some positive integer $N$.

**Proof:** Theorem 1.6.2 of [1] asserts that if there are no non-zero point derivations at $x_0$, then $x_0$ is an isolated point of $X$ in the norm topology. The result then follows from the preceding corollary. $\square$

## Part 2: Dales-Davie algebras

Let $X$ be a perfect compact subset of the complex plane and let $D^\infty(X)$ denote the set of infinitely differentiable functions on $X$. Suppose, too, that $(M_n)$ is a sequence of positive numbers satisfying $M_0 = 1$ and $\dfrac{M_{n+m}}{M_n M_m} \geq \dbinom{n+m}{m}$. Finally, let

$$D(X,M) = \{f \in D^\infty(X) : \|f\|_D = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_\infty}{M_n} < \infty\}.$$

With pointwise addition and multiplication, $D(X,M)$ is a normed algebra. We call such algebras Dales-Davie algebras. See [2] and [3] for examples and basic facts about Dales-Davie algebras, and [4], [5] and [13] for some results about endomorphisms of these algebras. We will also assume that a weight sequence $(M_n)$ is nonanalytic meaning that $\lim_{n\to\infty}(n!/M_n)^{1/n} = 0$. Suppose that $(M_n)$ is nonanalytic, $D(X,M)$ is a Banach algebra, and that the maximal ideal space of $D(X,M)$ is precisely $X$. In such cases, every unital endomorphism $T$ of $D(X,M)$ has the form $Tf(x) = f(\phi(x))$ for some continuous selfmap $\phi$ of $X$. As a final definition, a selfmap $\phi$ of a compact subset of the plane is called analytic if

$$\sup_k \left(\frac{\|\phi^{(k)}\|_\infty}{k!}\right)^{1/k} < \infty.$$

Before proceeding to our result about Riesz endomorphisms of Dales-Davie algebras, we point out the following easily verified fact.

**Theorem 2.1:** If $T$ is a unital Riesz endomorphism of a unital commutative semi-simple Banach algebra $B$ with connected maximal ideal space $X$, then $\{f : Tf = f\}$ is a one dimensional Banach algebra. That is, the eigenvalue 1 has multiplicity 1 and $Tf = f$ implies that $f$ is a constant.

**Lemma 2.2:** Let $X$ be a connected perfect compact subset of the complex plane, $(M_n)$ a non-analytic weight sequence and $D(X,M)$ a Banach algebra with maximal ideal space $X$. Suppose that $T$ is a Riesz endomorphism of $D(X,M)$ induced by the selfmap $\phi$ of $X$. If $x_0$ is the fixed point of $\phi$, then $|\phi'(x_0)| < 1$.

**Proof:** Assume that $D(X,M)$, $T$, $\phi$ and $x_0$ are as described. Let $M_{x_0} = \{f : f(x_0) = 0\}$ and $T_0 = T_{|M_{x_0}}$. The operator $T_0$ is a Riesz endomorphism of $M_{x_0}$. We first show that if $|\phi'(x_0)| = 1$, then for some positive integer $N$, $\lambda = 1$ is in the spectrum $\sigma(T_0^N)$ of $T_0^N$. Letting $\|S\|_{sp}$ denote the spectral radius of an operator $S$, we claim that if $|\phi'(x_0)| = 1$, then $\|T_0\|_{sp} = 1$. To this end, assume that $|\phi'(x_0)| = 1$ and let $f(x) = x - x_0$. Clearly $f \in D(X,M)$. Then for each positive integer $n$,

$$\|f\|_D\|T_0^n\| \geq \|T_0^n f\|_D \geq \|\phi_n - x_0\|_D \geq \frac{|\phi_n'(x_0)|}{M_1} = \frac{|\phi'(x_0)|^n}{M_1}.$$

Therefore, $\|T_0\|_{sp} = \lim \|T_0^n\|^{1/n} \geq |\phi'(x_0)| = 1$. On the other hand, since the set of eigenvalues of $T_0$ is closed under multiplication, $\sigma(T_0) \subseteq \{\lambda : |\lambda| \leq 1\}$. Hence, $\|T_0\|_{sp} = 1$. Also, since $T_0$ is a Riesz endomorphism every non-zero element in $\sigma(T_0)$ is an eigenvalue. Consequently there exists an eigenvalue $\lambda$ of $T_0$ of magnitude 1. Again using the fact that the eigenvalues

are closed under multiplication, and the fact that there are only finitely many eigenvalues of $T_0$ on the unit circle, we conclude that for some positive integer $N$, $1 \in \sigma(T_0^N)$. Thus, for some non-zero $u$ in $M_{x_0}$, $T_0^N u = u$. Further, the function $u$ is also an eigenvector of $T^N$ on $D(X, M)$. Hence from Theorem 2.1 $u$ is a constant function, and since $u \in M_{x_0}$, $u$ must be 0, a contradiction. Therefore if $T$ is a Riesz endomorphism of the Banach algebra $D(X, M)$, then $|\phi'(x_0)| < 1$. $\square$

Suppose that $X$ is a connected perfect compact subset of the complex plane and $(M_n)$ is a nonanalytic weight sequence. Suppose further that $D(X, M)$ is a Banach algebra with maximal ideal space $X$. It was shown in [5] that if $T$ is an endomorphism induced by an analytic selfmap $\phi$ of $X$ and if $\|\phi'\|_\infty < 1$, then $T$ is a compact endomorphism. The next theorem follows easily from this and Lemma 2.2.

**Theorem 2.3:** Let $X$ be a connected perfect compact subset of the complex plane and $(M_n)$ a nonanalytic weight sequence. If $D(X, M)$ is a Banach algebra with maximal ideal space $X$, then every Riesz endomorphism induced by an analytic selfmap $\phi$ of $X$ is power compact.

**Proof:** Suppose that $T$ is a Riesz endomorphism of $D(X, M)$ induced by an analytic selfmap $\phi$ of $X$ and $x_0$ is the fixed point of $\phi$. From Lemma 2.2, $|\phi'(x_0)| = c_1 < 1$. Let $0 < c_1 < c < 1$ and let $\mathcal{U}$ be a neighborhood of $x_0$ for which $|\phi'(t)| < c < 1$ for all $t \in \mathcal{U}$. Then there exists a positive integer $N$ such that $\phi_N(x) \in \mathcal{U}$ for all $x \in X$. Then for $n > N$,

$$|\phi_n'(x)| = |\phi'(\phi_{n-1}(x)) \cdots \phi'(\phi_{N+1}(x))\phi'(\phi_N(x)) \cdots \phi'(\phi(x))\phi'(x)|.$$

Hence if $n > N$, then for all $x \in X$, $|\phi_n'(x)| < c^{n-N}\|\phi'\|_\infty^N$. Thus $\|\phi_n'\|_\infty < 1$ for large $n$, whence $\phi$ induces compact endomorphism of $D(X, M)$ for large $n$. Therefore $T$ is power compact. $\square$

Further, the spectrum $\sigma(T)$ of a Riesz endomorphism $T$ of $D(X, M)$ is easy to determine in many cases. It is well known [3] that a sufficient condition for $D(X, M)$ to be a Banach algebra is that $X$ be uniformly regular, meaning that for all $z, w \in X$, there is a rectifiable arc in $X$ joining $z$ to $w$, and the metric given by the geodesic distance between the points of $X$ is uniformly equivalent to the Euclidean metric. Moreover, the proofs of Theorem 2.4 of [5] when $X = [0, 1]$ and Theorem 11 of [6] when $X$ is uniformly regular show that

$$\sigma(T) = \{(\phi'(x_0)^n : \text{n is a positive integer}\} \cup \{0, 1\}$$

holds whenever the non-zero spectrum contains only eigenvalues. Thus we have the following theorem.

**Theorem 2.4:** Let $X$ be a uniformly regular compact subset of the complex plane and $(M_n)$ a nonanalytic weight sequence. If $T$ is a Riesz endomorphism of $D(X, M)$ induced by $\phi$ and if $x_0$ is the fixed point of $\phi$, then

$$\sigma(T) = \{(\phi'(x_0)^n : \text{n is a positive integer}\} \cup \{0, 1\}.$$

Also, each nonzero element in $\sigma(T)$ has multiplicity 1.

Part 3. $C^1[0, 1]$

We next look at the Banach algebra $C^1[0, 1]$ for examples of Riesz endomorphisms which are not power compact. The technique in the proof of the theorem is a slight variation of Theorem 1.2.

**Theorem 3.1:** Suppose that $T$ is a unital endomorphism of $C^1[0, 1]$ induced by the selfmap $\phi$ of $[0, 1]$. Then $T$ is a Riesz endomorphism if, and only if, $\bigcap_{n=0}^{\infty} \phi_n([0, 1]) = \{x_0\}$ for some $x_0 \in [0, 1]$ and $\phi'(x_0) = 0$.

**Proof:** First suppose that $\phi$ induces a Riesz endomorphism $T$ of $C^1[0, 1]$, $\bigcap_{n=0}^{\infty} \phi_n([0, 1]) = \{x_0\}$ and $\phi'(x_0) \neq 0$. We aim to show that this leads to a contradiction. To this end, choose $x_n \in [0, 1]$ and $f_n \in C^1[0, 1]$ with $x_n \to x_0$, $f'_n(\phi(x_n)) = -1$, $f'_n(x_0) = 1$ and $\|f_n\|_{C^1} \leq 1 + \frac{1}{n}$. Suppose that $K$ is a compact operator on $C^1[0, 1]$. Then there exist $g \in C^1[0, 1]$ and a subsequence $\{f_{n_j}\}$ such that $Kf_{n_j} \to g$. As in the proof of Theorem 1.2,

$$\|f_{n_j}\|\|T - K\| \geq \|Tf_{n_j} - Kf_{n_j}\| \geq \|Tf_{n_j} - g\| - \|Kf_{n_j} - g\|,$$

and so

$$\|T - K\| \geq \limsup_j \|Tf_{n_j} - g\|.$$

Now

$$\limsup_j \|Tf_{n_j} - g\|_{C^1} =$$

$$\limsup_j \left( \sup_x |f_{n_j}(\phi(x)) - g(x)| + \sup_x |f'_{n_j}(\phi(x))\phi'(x) - g'(x)| \right).$$

First evaluating at $x_{n_j}$, we get

$$\limsup_j \|Tf_{n_j} - g\|_{C^1} \geq \limsup_j [0 + | - \phi'(x_{n_j}) - g'(x_{n_j})|] =$$

$$\limsup_j [|\phi'(x_{n_j}) + g'(x_{n_j})|] = |\phi'(x_0) + g'(x_0)|.$$

Then evaluating at $x_0$, we get

$$\limsup_j \|Tf_{n_j} - g\|_{C^1} \geq \limsup_j |\phi'(x_0) - g'(x_0)| = |\phi'(x_0) - g'(x_0)|.$$

Adding we get $\|T - K\| \geq |\phi'(x_0)|/2$. We recall that the compact operator $K$ is arbitrary.

Again, for each positive integer $m$, $T^m$ is a Riesz endomorphism which is induced by $\phi_m$; hence the preceding argument goes through for $T^m$. That is, $\|T^m - K\| \geq |\phi'(x_0)|^m/2$ for all positive integers $m$ and all compact operators $K$. Thus $\lim[\inf\{\|T^m - K\| : K \text{ is a compact operator}\}]^{1/m} \geq |\phi'(x_0)| > 0$. This is a contradiction to the assumption that $T$ is a Riesz endomorphism.

Conversely, assume that $\bigcap_{n=0}^{\infty} \phi_n([0,1]) = \{x_0\}$ and $\phi'(x_0) = 0$. Let $Lf = f(x_0)1$, a compact operator from $C^1[0,1]$ to $C^1[0,1]$. We show that $\lim_n \|T^n - L\|^{1/n} = 0$. Indeed, let $\varepsilon > 0$. Since $\bigcap_{n=0}^{\infty} \phi_n([0,1]) = \{x_0\}$ and $\phi'(x_0) = 0$, there exists a positive integer $N$ and a positive number $M$ such that $|\phi_n'(t)| < M\varepsilon^n$ for $n > N$, all $t \in [0,1]$. Then

$$|f(\phi_n(x)) - f(x_0)| \leq \|f'\|_\infty \|\phi_n'\|_\infty |x - x_0| < M_1\varepsilon^n$$

some $M_1$, all $x$, large $n$. Also

$$|f'(\phi_n(x))\phi_n'(x)| \leq \|f'\|_\infty |\phi_n'(x)| \leq M_2\varepsilon^n$$

for some $M_2$, all $x$, large $n$. Therefore, $\|T^n - L\| < (M_1 + M_2)\varepsilon^n$, which implies that $\lim_n \|T^n - L\|^{1/n} = 0$ whence $T$ is a Riesz operator by Theorem 52.2 of [11]. □

**Corollary 3.2:** There exist Riesz endomorphisms of $C^1[0,1]$ which are not power compact.

**Proof:** Every compact endomorphism of $C^1[0,1]$ has the form $Lf = f(x_0)1$ for some $x_0 \in [0,1]$. ([12], Theorem 2.3). As an example, if $\phi(x) = x^2/2$, then $\phi$ induces a Riesz endomorphism of $C^1[0,1]$ which is not power compact. □

## Part 4: Uniform algebras

Thus far we have examples of algebras for which every Riesz endomorphism is power compact, as well as an example of an algebra with a non-power compact Riesz endomorphism. As we now show, every Riesz endomorphism of the disk algebra or $H^{\infty}(\Delta)$, $\Delta$ the unit disk, is power compact. One might think that this was the case for every uniform algebra. However, in this section we also construct an example of a uniform algebra whose maximal ideal space is connected and which has a non-power compact Riesz endomorphism.

We will assume some knowledge of the standard theory of uniform algebras: for more details we refer the reader to [9].

Let $A$ be a uniform algebra with maximal ideal space $X$ and let $T$ be an endomorphism of $A$ induced by a selfmap $\phi$ of $X$. It is standard that $T$ is compact if and only if $\phi(X)$ is a norm (Gleason) compact subset of $X$ ([10, Theorem 1]). Note, however, that the 'if' part fails for more general Banach function algebras.

It follows from our earlier Theorem 1.2 that if $\phi$ induces a Riesz endomorphism $T$ of a uniform algebra with connected maximal ideal space $X$, then for some positive integer $N$, $\phi_N(X)$ is a Gelfand (and hence also Gleason) closed subset of a closed norm ball of radius less than 2. This ball is, of course, contained in exactly one Gleason part.

Recall that a uniform algebra $A$ is a *unique representing measure algebra* (*URM-algebra*) if every character of $A$ has a unique representing measure on the Shilov boundary of $A$. The disk algebra, $H^{\infty}(\Delta)$, and the trivial uniform algebras $C(X)$ are all examples of URM-algebras. It is standard that the Gleason parts of URM-algebras are either one point parts or else analytic disks. In particular, for these algebras, every closed Gleason ball of radius less than 2 is Gleason compact.

From the above discussion we now see immediately that if $A$ is a URM-algebra with connected maximal ideal space, then every Riesz endomorphism of $A$ is power-compact.

See [14] for related results and examples concerning compact endomorphisms.

In view of the above results, in order to construct non-power compact Riesz endomorphisms of uniform algebras, we should look at examples where the small closed Gleason balls are not Gleason compact. Klein [14] considered such uniform algebras for similar reasons. We begin by proving a lemma which gives a sufficient condition for an endomorphism of a uniform algebra to be Riesz.

**Lemma 4.1:** Let $A$ be a uniform algebra on a compact space $X$ and let $x_0 \in X$. Let $\phi$ be a self-map of $X$ which induces an endomorphism $T$ of $A$, and as usual let $\phi_n$ be the $n$th iterate of $\phi$. Using the norm (Gleason) distance in the maximal ideal space, set $C_n = \sup\{\|\phi_n(x) - x_0\| : x \in X\}$. Suppose that $C_n^{1/n} \to 0$ as $n \to \infty$. Then $T$ is a Riesz endomorphism of $A$.

**Proof:** Consider the distance from $T^n$ to the compact endomorphism $L$ given by $Lf = f(x_0)1$. For $f \in A$ and $x \in X$ we have $(T^n f - Lf)(x) = f(\phi_n(x)) - f(x_0)$ and it then follows from this that $\|T^n - L\| \leq C_n$. Thus if $C_n^{1/n} \to 0$ as $n \to \infty$, then $\|T^n - L\|^{1/n} \to 0$ as $n \to \infty$, and the result follows. □

This condition is far from necessary, as is shown by, for example, the compact endomorphism of the disk algebra induced by the self-map $\phi(z) = z/2$.

We now use Lemma 4.1 to construct a non-power compact Riesz endomorphism of a Banach algebra of analytic functions on the unit ball of a Banach space.

For our example let $\mathcal{X}$ denote the closed unit ball of the complex Banach space $E = \ell^\infty(\mathbf{N}^2)$, with the weak * topology. Then $\mathcal{X}$ is compact and metrizable. Elements of $\mathcal{X}$ will be denoted by $x = (x_{j,k})_{j,k=1}^\infty$. Next define $\mathcal{A}$ as the uniform algebra on $\mathcal{X}$ generated by the co-ordinate projections $p_{j,k}$. Then $\mathcal{A}$ is a subalgebra of the uniform algebra of all continuous functions on $\mathcal{X}$ which are disk algebra functions in each variable separately.

We consider the norm metric $d_\infty$ on the Banach space $E$, and also the Gleason distance (from the norm of $\mathcal{A}^*$) on $\mathcal{X}$. Further we denote by $x^0$ the zero element of $E$: obviously $x^0 \in \mathcal{X}$.

Although we do not use this fact, it is easy to show that the maximal ideal space of $\mathcal{A}$ is $\mathcal{X}$. Indeed, every character is determined by what it does to all of the coordinate functionals, and must agree at all of these with some evaluation character at a point of $\mathcal{X}$.

We start with a lemma to help us estimate the Gleason distance from $x^0$.

**Lemma 4.2:** Let $\mathcal{A}$ and $\mathcal{X}$ be as described.

(i) If $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$, then the selfmap $\psi$ of $\mathcal{X}$ defined by $\psi(x) = \alpha x$ induces an endomorphism of $\mathcal{A}$.

(ii) For each $x \in \mathcal{X}$, the Gleason distance from $x$ to $x^0$ is at most $2d_\infty(x, x^0)$.

**Proof:** (i) For all $j, k$ we have $p_{j,k} \circ \psi = \alpha p_{j,k} \in \mathcal{A}$. Thus the closed subalgebra $\{f \in C(\mathcal{X}) : f \circ \psi \in \mathcal{A}\}$ of $C(\mathcal{X})$ contains all the $p_{j,k}$ and hence all of $\mathcal{A}$, as required.

(ii) Let $x \in \mathcal{X}$ and assume that $x \neq x^0$. Set $R = 1/d_\infty(x, x^0)$. Let $f$ be a function in the unit ball of the dense subalgebra of $A$ generated by 1 and the coordinate projections $p_{j,k}$ and consider the function $g$ defined on the closed unit disk by $g(z) = f(zRx) - f(x^o)$. This is simply a polynomial in $z$ which vanishes at 0, and $\|g\|_\infty \leq 2\|f\|_\infty$. Note the norm of $g$ is taken on the closed unit disc, so by Schwartz's Lemma we must have $|g(z)| \leq 2\|f\|_\infty |z|$ for $|z| \leq 1$. In particular, setting $z = 1/R$ gives us $|f(x) - f(x^0)| \leq 2\|f\|_\infty / R = 2d_\infty(x, x^0)$. The result now follows. □

**Theorem 4.3:** With $\mathcal{A}$ and $\mathcal{X}$ as above, there is a Riesz endomorphism of $\mathcal{A}$ which is not power compact.

**Proof:** We define a certain 'weighted shift' $\phi$ on $\mathcal{X}$ and show that this induces an endomorphism with the desired properties.

For $x \in \mathcal{X}$, we define $\phi$ by

$$(\phi(x))_{j,k} = x_{j,k+1}/(k+1).$$

Note that

$$d_\infty(\phi_n(x), x^0) \leq 1/(n+1)! \qquad (*)$$

for all $n$ and all $x \in \mathcal{X}$, and in particular $\phi$ is a selfmap of $\mathcal{X}$.

We next show that $\phi$ induces an endomorphism of $\mathcal{A}$. By definition of $\phi$ we have, for all $j, k$, $p_{j,k} \circ \phi = p_{j,k+1}/(k+1)$. Thus (as for $\psi$ above) $f \circ \phi \in \mathcal{A}$ for all $f \in \mathcal{A}$. Let $T$ be the endomorphism induced by $\phi$. It follows easily from the preceding two lemmas and $(*)$ that $T$ is Riesz. To show that $T$ is not power compact, let $n \in \mathbf{N}$ and note that for all $j$, $(T^n p_{j,1}) = p_{j,n+1}/(n+1)!$. Since $(T^n p_{j,1})_{j=1}^\infty$ has no convergent subsequence, $T^n$ is not compact for all $n$, whence the Rieesz endomorphism $T$ is not power compact. □

This algebra was also used by Klein in [14]. In addition, a variety of related algebras were considered by Galindo, Gamelin, and Lindström [8] in relation to weakly compact homomorphisms. It is easy to see from their work that weakly compact endomorphisms need not be Riesz.

Since weakly compact endomorphisms of the disk algebra are always compact [7] it follows that there are Riesz endomorphisms of the disk algebra which are not weakly compact. Such endomorphisms are, of course, power compact. It is easy to see, however, that the Riesz endomorphism constructed in Theorem 4.3 has the property that no iterate of it is weakly compact.

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